

# Gravitational Interactions and Fine-Structure Constant

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## Abstract

Electromagnetic and gravitational central-field problems are studied with relativistic quantum mechanics on curved space-time backgrounds. Corrections to the transition current are identified. Analogies of the gravitational and electromagnetic spectra suggest the definition of a gravitational fine-structure constant. The electromagnetic and gravitational coupling constants enter the Einstein–Hilbert–Maxwell Lagrangian. We postulate that the variational principle holds with regard to a global dilation transformation of the space-time coordinates. The variation suggests is consistent with a functional relationship of the form  $\alpha_{\text{QED}} \propto (\alpha_G)^{1/2}$ , where  $\alpha_{\text{QED}}$  is the electrodynamic fine-structure constant, and  $\alpha_G$  its gravitational analogue.

## 1 Introduction

If we are ever to gain a better understanding of the relationship of gravitational interactions and electrodynamics in the quantum world, then a very practical approach is to try to solve a number of important example problems in gravitational theory, whose solution is known in electromagnetic theory, to try to generalize the approach to the gravitational analogue, and to compare. In order to proceed, it is not necessarily required to quantize space-time itself [1]. Indeed, the formulation of quantum mechanics on curved-space backgrounds in itself constitutes an interesting problem [2–6].

A priori, one might think that the simple substitution  $\partial/\partial x^i \rightarrow \nabla_i$  is the Schrödinger equation might suffice. Here,  $\partial/\partial x^i$  is the  $i$ th partial derivative with respect to the  $i$ th spatial coordinate, whereas  $\nabla_i$  is the  $i$ th covariant derivative. However, this naive approach is destined to fail; the gravitational theory of Einstein and Hilbert inherently is a relativistic theory, and the only way to describe quantum particles on curved space-times is to start from a fully relativistic wave function. The Dirac equation

$$(i\gamma^\mu \partial_\mu - m) \psi(x) = 0 \quad (1)$$

generalizes as follows to a curved space-time background [2–6],

$$(i\gamma^\mu(x)(\partial_\mu - \Gamma_\mu) - m) \psi(x) = 0. \quad (2)$$

The Dirac algebra [7–9] needs to be generalized to the local metric  $g^{\mu\nu}(x)$ ,

$$\{\gamma^\mu(x), \gamma^\nu(x)\} = 2g^{\mu\nu}(x), \quad \sigma^{\mu\nu}(x) = \frac{i}{2}[\gamma^\mu(x), \gamma^\nu(x)]. \quad (3)$$

The spin connection matrix  $\Gamma_\mu$  is given as

$$\Gamma_\mu = -\frac{i}{4} g_{\rho\alpha}(x) \left( \frac{\partial b_\nu^\beta(x)}{\partial x^\mu} a^\alpha_\beta(x) - \Gamma^\alpha_{\nu\mu} \right) \sigma^{\rho\nu}(x), \quad (4)$$

where repeated indices are summed. Finally, the  $a$  and  $b$  coefficients belong to the square root of the metric,

$$\gamma_\rho(x) = b_\rho^\alpha(x) \gamma_\alpha, \quad \gamma^\alpha(x) = a^\alpha_\rho(x) \gamma^\rho, \quad (5)$$

where the  $\gamma^\alpha$  are the flat-space Dirac matrices, which are preferentially used in the Dirac representation [1, 9–12]. The Christoffel symbols are  $\Gamma^\alpha_{\nu\rho} \equiv \Gamma^\alpha_{\nu\rho}(x)$ .

## 2 Central-Field Problem

### 2.1 Foldy–Wouthuysen Method

The Foldy–Wouthuysen method [13,14] is a standard tool for the extraction of the physical, nonrelativistic degrees of freedom, from a fully relativistic Dirac theory. The general paradigm is as follows: The positive and negative energy solutions of a (generalized) Dirac equation are intertwined in the fully relativistic formalism. One has to separate the upper and lower spinors in the bispinor solution, and in order to do so, one eliminates the “off-diagonal couplings” of the upper and lower spinor components order by

order in some perturbative parameters, possibly, using iterated (unitary) transformations.

For the plain free Dirac Hamiltonian, a standard method exists to all orders in perturbation theory, while for more difficult problems, one manifestly has to resort to a perturbative formalism [13, 14]. A suitable expansion parameter in a general case is the particle's momentum operator. Let us consider a space-time metric of the form

$$\bar{g}_{\mu\nu} = \text{diag} (w^2(r), -v^2(r), -v^2(r), -v^2(r)) . \quad (6)$$

The Schwarzschild metric in isotropic coordinates (see Sec. 43 of Chap. 3 of Ref. [15]), involves the Schwarzschild radius  $r_s$ ,

$$\begin{aligned} w &= \left(1 - \frac{r_s}{4r}\right) \left(1 + \frac{r_s}{4r}\right)^{-1} = \frac{4r - r_s}{4r + r_s} \approx 1 - \frac{r_s}{2r} , \\ v &= \left(1 + \frac{r_s}{4r}\right)^2 \approx 1 + \frac{r_s}{2r} , \quad \frac{w}{v} = \frac{16 r^2 (4r - r_s)}{(4r + r_s)^3} \approx 1 - \frac{r_s}{r} . \end{aligned} \quad (7)$$

The Schwarzschild radius reads as  $r_s = 2 G M$ , where  $G$  is Newton's gravitational constant, and  $M$  is the mass of the planet (or "black hole"). The Hamiltonian or time translation operator is necessarily "noncovariant" in the sense that the time coordinate needs to be singled out. If we insist on using the time translation with respect to the time coordinate  $dt$  in the metric  $ds^2 = w^2(r) dt^2 - v^2(r) d\vec{r}^2$  and bring the Hamiltonian into Hermitian form [see Ref. [16] and Eqs. (9)–(13) of Ref. [10]], then we obtain

$$H_{\text{DS}} = \frac{1}{2} \left\{ \vec{\alpha} \cdot \vec{p}, \left(1 - \frac{r_s}{r}\right) \right\} + \beta m \left(1 - \frac{r_s}{2r}\right) , \quad (8)$$

where  $\alpha^i = \gamma^0 \gamma^i$  is the Dirac  $\alpha$  matrix (we here use the Dirac representation). The Foldy–Wouthuysen transformed Dirac–Schwarzschild Hamiltonian is finally obtained as [10]

$$\begin{aligned} H_{\text{FW}} &= \beta \left( m + \frac{\vec{p}^2}{2m} - \frac{\vec{p}^4}{8m^3} \right) - \beta \frac{m r_s}{2r} \\ &+ \beta \left( -\frac{3r_s}{8m} \left\{ \vec{p}^2, \frac{1}{r} \right\} + \frac{3\pi r_s}{4m} \delta^{(3)}(\vec{r}) + \frac{3r_s}{8m} \frac{\vec{\Sigma} \cdot \vec{L}}{r^3} \right) . \end{aligned} \quad (9)$$

The parity-violating terms obtained in Refs. [16, 17] are spurious.

## 2.2 Transition Current

As we couple the Dirac–Schwarzschild Hamiltonian (8) to an electromagnetic field (see Fig. 1), it is clear that the transition current in the interac-

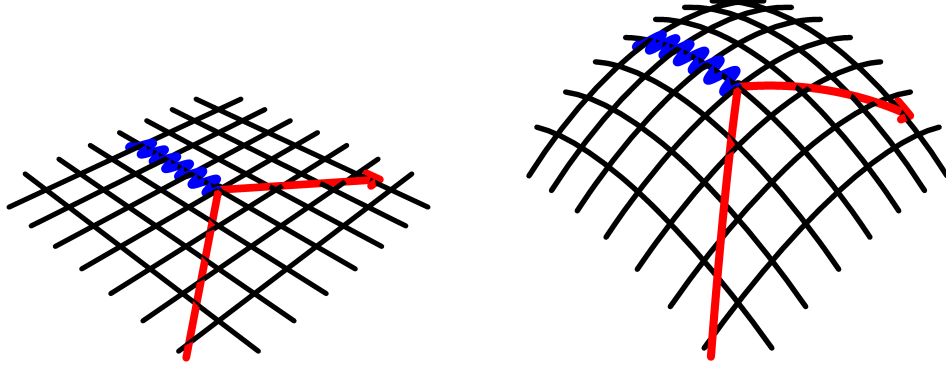


Figure 1: The flat-space photon emission vertex (left figure) is promoted to a curved-space vertex (right figure) in general relativity. The curved background leads to higher-order corrections to the transition current, which are summarized, for the Schwarzschild metric, in Eq. (12).

tion Hamiltonian is  $H_{\text{int}} = -\vec{j} \cdot \vec{A}$ . takes the form

$$j^i = \frac{1}{2} \left\{ 1 - \frac{r_s}{r}, \alpha^i \exp(i\vec{k} \cdot \vec{r}) \right\}. \quad (10)$$

We now employ the multipole expansion

$$\alpha^i \exp(i\vec{k} \cdot \vec{r}) \approx \alpha^i + \alpha^i (i\vec{k} \cdot \vec{r}) - \frac{1}{2} \alpha^i (\vec{k} \cdot \vec{r})^2 \quad (11)$$

A unitary transformation with the same generators are used for the Dirac–Schwarzschild Hamiltonian then yields the result [10],

$$\begin{aligned} j_{\text{FW}}^i = & \frac{p^i}{m} - \frac{p^i \vec{p}^2}{2m} - \frac{i}{2m} (\vec{k} \times \vec{\sigma})^i + \frac{1}{2} \left\{ \frac{p^i}{m}, (i\vec{k} \cdot \vec{r}) \right\} \\ & - \frac{1}{4} \left\{ (\vec{k} \cdot \vec{r})^2, \frac{p^i}{m} \right\} + \frac{1}{2m} (\vec{k} \cdot \vec{r}) (\vec{k} \times \vec{\sigma})^i \\ & - \frac{3}{4} \left\{ \frac{p^i}{m}, \frac{r_s}{r} \right\} + \frac{r_s}{2r} \frac{(\vec{\sigma} \times \vec{r})^i}{m r^2} - \frac{1}{2} \left\{ (i\vec{k} \cdot \vec{r}), \left\{ \frac{p^i}{m}, \frac{r_s}{r} \right\} \right\} \\ & + \frac{3ir_s}{4r} \frac{(\vec{k} \times \vec{\sigma})^i}{m} + \frac{1}{4} \left\{ \frac{r_s}{r} (i\vec{k} \cdot \vec{r}), \frac{p^i}{m} \right\}. \end{aligned} \quad (12)$$

This result contains a gravitational kinetic correction, and gravitational corrections to the magnetic coupling, in addition to the known multipole and retardation corrections [14, 18].

## 2.3 Spectrum

The bound-state spectrum resulting from the Hamiltonian (8) has recently been evaluated as [12],

$$\begin{aligned}
E_{n\ell j} = & -\frac{\alpha_G^2 m_e c^2}{2n^2} + \alpha_G^4 m_e c^2 \left( \frac{15}{8n^4} \right. \\
& \left. - \frac{(7j+5)\delta_{\ell,j+1/2}}{(j+1)(2j+1)n^3} - \frac{(7j+2)\delta_{\ell,j-1/2}}{j(2j+1)n^3} \right) \\
= & -\frac{\alpha_G^2 m_e c^2}{2n^2} + \frac{\alpha_G^4 m_e c^2}{n^3} \left( \frac{15}{8n} - \frac{14\kappa+3}{2|\kappa|(2\kappa+1)} \right),
\end{aligned} \tag{13}$$

where  $\ell$  is the orbital angular momentum,  $j$  is the total angular momentum of the bound particle, and  $\kappa$  is the (integer) Dirac angular quantum number,

$$\kappa = 2(\ell - j)(j + 1/2) = (-1)^{j+\ell+1/2} \left( j + \frac{1}{2} \right). \tag{14}$$

For a bound electron-proton system, the coupling constant entering the gravitational spectrum given in Eq. (13) reads as

$$\alpha_G = \frac{G m_e m_p}{\hbar c} = 3.21637(39) \times 10^{-42}. \tag{15}$$

The coupling  $\alpha_G$  is much larger than for particles bound to macroscopic objects. By contrast, the electrodynamic coupling parameter

$$\alpha_{\text{QED}} = \frac{e^2}{4\pi\hbar\epsilon_0 c} \approx \frac{1}{137.036} \tag{16}$$

is just the fine-structure constant.

## 3 Global Dilation Transformation

### 3.1 Lagrangian

The analogy of the leading (Schrödinger) term in Eq. (13) for the non-relativistic contribution to the bound-state energy (under the replacement  $\alpha_G \rightarrow \alpha_{\text{QED}}$ ) may encourage us to look for connections of gravitational and electromagnetic interactions on a more global scale, possibly, using scaling transformations [19]. Indeed, the first attempts to unify electromagnetism with gravity are almost 100 years old [20,21]. Let us apply a scaling transformation to the boson and fermion fields,

$$A^\mu \rightarrow \lambda A^\mu, \quad A_\mu \rightarrow \lambda A_\mu, \quad \psi \rightarrow \lambda \psi, \tag{17}$$

combined with a transformation of the coordinates,

$$x^\mu \rightarrow \lambda^{-1/2} x^\mu, \quad x_\mu \rightarrow \lambda^{-1/2} x_\mu, \quad (18)$$

and of the metric

$$g_{\mu\nu} \rightarrow \lambda g_{\mu\nu}, \quad g^{\mu\nu} \rightarrow \lambda^{-1} g^{\mu\nu}, \quad (19)$$

Under this transformation, the space-time intervals, the integration measure, the Ricci tensor  $R_{\mu\nu}$  and the curvature scalar  $R$ , transform as follows,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g^{\mu\nu} dx_\mu dx_\nu \rightarrow ds^2, \quad (20a)$$

$$d^4x = d^4x \rightarrow \frac{d^4x}{\lambda^2}, \quad \det g = \det g_{\mu\nu} \rightarrow \lambda^4 \det g, \quad (20b)$$

$$R_{\mu\nu} \rightarrow \lambda R_{\mu\nu}, \quad R = g^{\mu\nu} R_{\mu\nu} \rightarrow R. \quad (20c)$$

The Einstein–Maxwell Lagrangian, with a coupling to the fermion terms, is given as

$$S = \int d^4x \sqrt{-\det g} \left\{ \frac{R}{16\pi G} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(x) [i\bar{\gamma}^\mu (\nabla_\mu - e A_\mu) - m] \psi(x) \right\}. \quad (21)$$

It transforms into

$$S' = \int \frac{d^4x}{\lambda^2} \sqrt{-\lambda^4 \det g} \left\{ \frac{R}{16\pi G} - \frac{\lambda^2}{4} F^{\mu\nu} F_{\mu\nu} + \lambda^2 \bar{\psi}(x) \left[ i\lambda^{-1/2} \bar{\gamma}^\mu \left( \lambda^{1/2} \nabla_\mu - e \lambda A_\mu \right) - m \right] \psi(x) \right\}, \quad (22)$$

which can be rearranged into

$$S'' = \frac{S'}{\lambda^2} = \int d^4x \sqrt{-\det g} \left\{ \frac{R}{16\pi G \lambda^2} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(x) \left[ i\bar{\gamma}^\mu \left( \nabla_\mu - e \lambda^{1/2} A_\mu \right) - m \right] \psi(x) \right\}. \quad (23)$$

The Lagrangian  $S''$  is the same  $S$ , but with scaled coupling constants,

$$G \rightarrow \lambda^2 G, \quad e^2 \rightarrow \lambda e^2. \quad (24)$$

This scaling suggests a deeper connection of the coupling constants of electromagnetic and gravitational interactions, which is explored in further detail in Ref. [19].

### 3.2 Coupling Constants

If we assume that the scaling (24) holds globally, with the current Universe “picking” a value of  $\lambda$ , then this scaling might suggest a relationship of the type

$$\alpha_{\text{QED}}^2 \propto e^4 \propto \lambda^2 \propto G. \quad (25)$$

Indeed, as discussed in Ref. [19], a relationship of the type  $\alpha_{\text{QED}} \propto \sqrt{G}$  is otherwise suggested by string theory; the rough analogy being that gravitational interactions in string theory correspond to “closed” strings while electromagnetic interactions correspond to “open” strings. The product of two “open” string amplitudes is proportional to  $e^2 \propto g_0^2 \propto \alpha_{\text{QED}}$ , while the “closed”-string amplitude is proportional to  $\kappa \propto g_c \propto \sqrt{G}$ . According to Eq. (3.7.17) of Ref. [22], the proportionality

$$g_o^2 \propto g_c \quad \Leftrightarrow \quad \alpha_{\text{QED}}^2 \propto \sqrt{G} \quad (26)$$

therefore is suggested by string theory. A simple analytic form of the proportionality factor in the relationship  $\alpha_{\text{QED}}^2 \propto \sqrt{G}$  has recently been given in Eq. (8) of Ref. [1].

## 4 Conclusions

We have performed an analysis of the gravitationally coupled Dirac equation in the curved space-time surrounding a central gravitating object, which is described by the (static) Schwarzschild metric. The Foldy–Wouthuysen method leads to gravitational zitterbewegung terms and the gravitational spin-orbit coupling, which is also known as the Fokker precession term. In a curved space-time, the photon emission vertex receives additional corrections due to the curved background, which can be given, within the multipole expansion and for a conceptually simple background metric (e.g., the Schwarzschild metric), in closed analytic form (at least for the first terms of the multipole and retardation expansion). The gravitational bound states display a certain analogy for the gravitational as compared to the electromagnetic (Schrödinger) central-field problem. Based on this analogy, one may explore possible connections of the gravitational and electromagnetic coupling constants, based on scaling arguments. Such a scaling transformation gives additional support for the relationship  $\alpha_{\text{QED}}^2 \propto \sqrt{G}$ , which has been suggested by string theory [22].

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